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THE STRESS INTENSITY FACTOR FOR AN EXTERNAL ELLIPTICAL CRACK

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Abstract—The paper reveals an error in the well-known formula by Kassir and Sih [*Three-dimensional Crack Problems* (Noordhoof, Leyden, 1985)] related to the stress intensity factor for an external elliptical crack in a three-dimensional medium. An elementary derivation of the correct formula is given. Some numerical results are presented which illustrate the difference between the formulae.

THEORY

Consider an elastic space weakened by an external elliptical crack in the plane $z = 0$. Let a and b be the major and the minor semiaxes of the ellipse. According to Kassir and Sih[1], the normal stress distribution due to an axial force P^∞, acting at infinity in the z-direction, has the form

$$\sigma_{zz} = \frac{P^\infty}{2\pi ab\left[1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}\right]^{1/2}} \tag{1}$$

where x, y, z are Cartesian coordinates. It is then necessary to perform some elementary transformations. Introducing polar coordinates ρ and ϕ, as shown in Fig. 1, let $x = \rho\cos\phi$ and $y = \rho\sin\phi$. Furthermore, let

$$c(\phi) = \frac{ab}{[a^2\sin^2\phi + b^2\cos^2\phi]^{1/2}}. \tag{2}$$

By using this notation, eqn (1) can be rewritten as

$$\sigma_{zz} = \frac{c(\phi)P^\infty}{2\pi ab[c^2(\phi) - \rho^2]^{1/2}}. \tag{3}$$

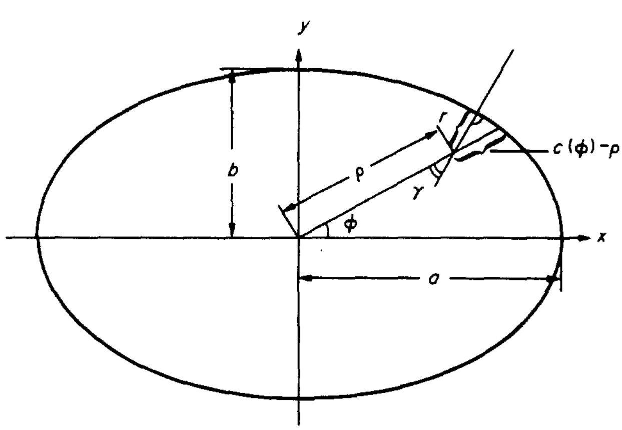

Fig. 1.

If the stress intensity factor is defined by

$$k = \lim_{\rho \to c(\phi)} \{\sigma_{zz}[2(c(\phi) - \rho)]^{1/2}\} \tag{4}$$

then the following expression results from eqn (3)

$$k = \frac{P^{\infty}\sqrt{(c(\phi))}}{2\pi ab} = \frac{P^{\infty}}{2\pi\sqrt{(ab)}(a^2 \sin^2\phi + b^2\cos^2\phi)^{1/4}}. \tag{5}$$

This result corresponds to the formula given by Kassir and Sih[1], obtained on the assumption of an approach to the crack border along a direction normal to the crack front. The definition of the stress intensity factor given by eqn (4) implies approach along the polar radius, and gives the same result. This means that either the stress intensity factor does not depend on the direction of approach or that the formula of Kassir and Sih is incorrect. To follow are some considerations which clarify this apparent discrepancy.

The correct definition of the stress intensity factor should imply the normal approach to the crack border, and is, according to Kassir and Sih[1]

$$K = \lim_{r \to 0}[\sigma_{zz}\sqrt{(2r)}]. \tag{6}$$

From the geometrical considerations (Fig. 1), it is clear that $r = [c(\phi) - \rho]\cos\gamma$ where γ is the angle between the polar radius and the normal to the crack front. Comparison of eqns (4) and (6) leads to the conclusion that

$$K = k\sqrt{(\cos\gamma)}. \tag{7}$$

Analytical geometry provides the following expression for cos γ

$$\cos\gamma = \frac{c(\phi)}{[c^2(\phi) + c'^2(\phi)]^{1/2}} \tag{8}$$

where

$$c'(\phi) = \frac{dc(\phi)}{d\phi}.$$

Substitution of eqn (2) into eqn (8) yields

$$\cos\gamma = \frac{a^2\sin^2\phi + b^2\cos^2\phi}{[a^4\sin^2\phi + b^4\cos^2\phi]^{1/2}} \tag{9}$$

and the final expression for the stress intensity factor takes the form

$$K = \frac{P^{\infty}}{2\pi\sqrt{(ab)}}\left[\frac{a^2\sin^2\phi + b^2\cos^2\phi}{a^4\sin^2\phi + b^4\cos^2\phi}\right]^{1/4}. \tag{10}$$

Introducing the local radius of curvature R, where

$$R = \frac{1}{ab}\left[\frac{a^4\sin^2\phi + b^4\cos^2\phi}{a^2\sin^2\phi + b^2\cos^2\phi}\right]^{3/2}.$$

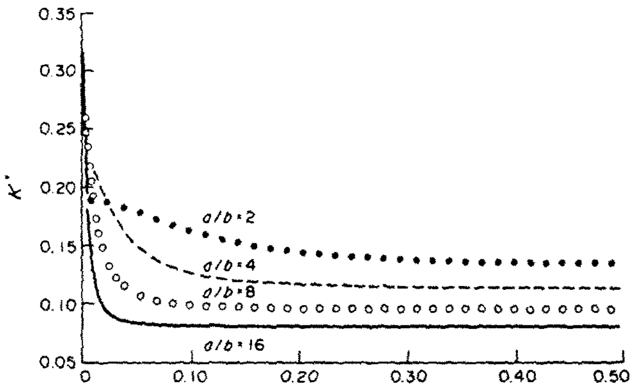

Fig. 2.

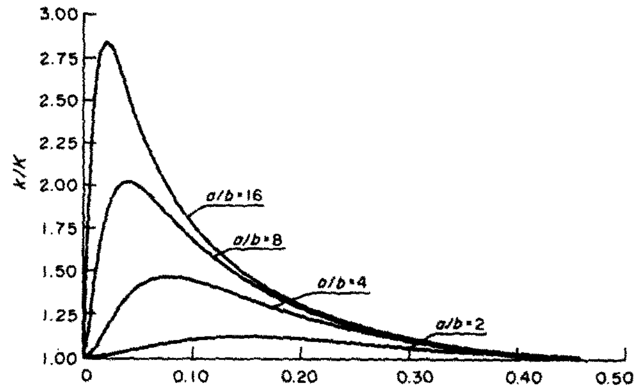

Fig. 3.

Equation (10) can be rewritten as

$$K = \frac{P^{\infty}}{2\pi(ab)^{2/3}R^{1/6}}. \tag{11}$$

Figure 2 presents a plot of the dimensionless $K^* = K(ab)^{3/4}/P^{\infty}$ vs the ratio ϕ/π for $a/b = 2, 4, 8, 16$.

CONCLUSION

It has been proven that eqn (5) for the stress intensity factor, given by Kassir and Sih[1], does not correspond to their own definition [eqn (6)] but represents the limit due to eqn (4) which implies approach to the crack border along the polar radius. The correct formula is given by eqn (10). It is of interest to compare the numerical results due to eqns (5) and (10). Figure 3 presents a plot of the ratio k/K against the polar angle for the values of $a/b = 2, 4, 8, 16$. As one can see, the discrepancy between eqns (5) and (10) increases with the eccentricity and can be quite significant. Similar corrections should be made in the other formulae in (Kassir and Sih[1]) treating the stress intensity factor for an elliptical crack under various conditions, including the formulae for internal elliptical cracks.

REFERENCE

1. M. K. Kassir and G. Sih, *Three-dimensional Crack Problems*. Noordhoof, Leyden (1975).